A Note on Intersecting and Fluctuating Solitons in 4D Noncommutative Field Theory

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Abstract

We examine the intersections, fluctuations and deformations of codimension two solitons in field theory on noncommutative $\mathbb{R}^4$, in the limit of large noncommutativity. We find that holomorphic deformations are zero modes of flat branes, and we show that there is a zero mode localized at the intersection of two solitons.

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1 Introduction

Field theories on noncommutative spaces possess a surprisingly rich dynamical structure (see for instance [1]-[4].) Recently, soliton solutions of scalar field theory on noncommutative $\mathbb{R}^n$ (nD NCFT) have been found in the limit of large noncommutativity [5]. The results of [5] have been extended to gauge theories in [6]-[14].

These results are particularly exciting in light of the application to D-branes in string theory [15]-[20].

In this paper we will study the fluctuations of a two-dimensional soliton (2-brane) in scalar 4D noncommutative field theory and begin the study of intersecting 2-branes in this theory. On the geometrical level, a plane in four dimensions can be deformed into a curved minimal area surface, and two intersecting planes can be deformed into a single smooth minimal area surface. In string theory, the latter phenomenon is related to a zero-mode which appears at the intersection of two D-branes [21].

The purpose of these notes is to:

1. Study the small deformations of planar 2-branes.
2. Identify the classical solutions corresponding to intersecting 2-branes.
3. Identify the zero-mode corresponding to a deformation into a smooth surface.

In principle, the zero mode at the intersection of two 2-branes might not correspond to an exact flat direction because of the existence of a quartic (or higher) potential. We will not explore this issue directly but we will study deviations from the linear equations of motions in the case of small fluctuations of a flat 2-brane.

Following [5], we will work in the large noncommutativity limit but include the kinetic energy to first order.

The paper is organized as follows. In section (2) we review the geometry of the deformation of intersecting planes. In section (3) we review the constructions of [5] and study the deformation modes of a single 2-brane in 4D scalar NCFT. We will show that half of the deformation modes correspond to deformations of the flat 2-brane into a holomorphic curve embedded in $\mathbb{R}^4$. The other half correspond to anti-holomorphic fluctuations. In section (4) we will study the holomorphic and anti-holomorphic fluctuations to higher order. In

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particular cases, we obtain deviations from pure holomorphicity at 7th order! In section (5) we describe the solution corresponding to two intersecting branes and study the zero-modes that correspond to their deformations. In section (6) some extensions to the case of multiple branes and more dimensions are discussed. We also briefly comment on the situation with $U(\infty)$ gauge fields.

2 Classical Geometry

We will consider surfaces in $\mathbb{R}^4$ that can be described by a holomorphic equation when $\mathbb{R}^4$ is identified with $\mathbb{C}^2$. Such surfaces have a minimal area in the sense that small deformations of the surface, keeping the boundary conditions at infinity intact, never decrease the area. Let the coordinates be:

$$z_k \equiv x_k + iy_k, \qquad k = 1, 2.$$

Consider first a surface that spans the z_2-direction and is given by the equation $z_1 = 0$. Small holomorphic deformations are described by $z_1 = \epsilon f(z_2)$ with $f(z) = \sum_{n=0}^{\infty} c_n z^n$ a holomorphic function.

Now consider adding a second surface spanning the z_1 direction, with the equation $z_2 = 0$. The two surfaces can be represented together by the equation $z_1 z_2 = 0$. This reducible surface can be deformed into a smooth irreducible surface given by $z_1 z_2 = \zeta$ where ζ is a complex number. This is the only holomorphic deformation of the singular surface $z_1 z_2 = 0$ that preserves the boundary conditions $z_1 \to 0$ as $|z_2| \to \infty$ and $z_2 \to 0$ as $|z_1| \to \infty$.

In this case, we see that the possible deformations are given by $z_1 = \epsilon f(z_2)$ where $f(z) = \sum_{n=-1}^{\infty} c_n z^n$ is allowed to have a simple pole at $z = 0$. More generally, if we add r surfaces given by the planes $z_2 = \xi_j$ ($j = 1 \ldots r$), we can have deformations $z_1 = \epsilon f(z_2)$ where f is a meromorphic function that is allowed to have simple poles at $\xi_1, \ldots, \xi_r$. If we add a surface $z_2 = 0$ with multiplicity k, then $f(z)$ is allowed to have a pole of kth order at the origin.

3   A Single Brane and its Fluctuations

In this section we will construct solitons of noncommutative scalar field theory along the lines of [5].

3.1   The Soliton

Let us review the construction of [5] for a single codimension-2 brane in the theory with action:

$$\int [(\partial_\mu \Phi)^2 + V(\Phi)].$$

Here:

$$V(\lambda) = \sum_{n=2}^{\infty} a_n \lambda^n, \qquad V(\Phi) = a_2 \Phi \star \Phi + a_3 \Phi \star \Phi \star \Phi + \cdots$$

We take spacetime to be commutative and define the $\star$-product as:

$$\Phi \star \Psi \equiv \Phi e^{\frac{i\theta}{2} \overset{\leftarrow}{\frac{\partial}{\partial x_1}} \overset{\rightarrow}{\frac{\partial}{\partial y_1}} - \frac{i\theta}{2} \overset{\leftarrow}{\frac{\partial}{\partial y_1}} \overset{\rightarrow}{\frac{\partial}{\partial x_1}}} \Psi$$

So that:

$$x_1 \star y_1 - y_1 \star x_1 = i\theta.$$

We take the limit $\theta \to \infty$. After a rescaling of the coordinates, the kinetic term is of order $1/\theta$ and can be neglected. For now, the x_2, y_2 coordinates are still commutative.

We set $z_1 = x_1 + iy_1$ and define a Hilbert space $\mathcal{H}_1$ with the harmonic oscillator basis, $|n\rangle$ for $n = 0, 1, \ldots$, such that $\hat{a}_1^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}_1 |n\rangle = \sqrt{n}|n-1\rangle$. If Φ is a function of x_1 and y_1, the Weyl formula transforms it into an operator on this Hilbert space:

$$\hat{\Phi} \equiv \frac{1}{2\pi} \int d^2\zeta \, \Phi(z_1, \overline{z}_1) e^{i\zeta \overline{z}_1 - i\overline{\zeta} z_1}.$$

Then $z_1 \to \sqrt{2\theta}\hat{a}_1$ and $\overline{z}_1 \to \sqrt{2\theta}\hat{a}_1^\dagger$. From now on, $\Phi, \Psi, \ldots$ will denote ordinary functions and $\hat{\Phi}, \hat{\Psi}, \ldots$ will denote the corresponding operators.

Let us assume that $V(\Phi)$ has a minimum at $\lambda \neq 0$. One can then construct a soliton by setting:

$$\hat{\Phi} = \lambda \hat{P}, \qquad \hat{P}^2 = \hat{P}.$$

The operator $\hat{\Phi}$ satisfies $V(\hat{\Phi}) = V(\lambda)\hat{P}$ and hence $V'(\hat{\Phi}) = 0$. The corresponding (Weyl transformed) solution, Φ, is constant in the z_2 direction. For any unitary operator, $\hat{U}$, V'$(\hat{U}^{\dagger}\hat{\Phi}\hat{U})$ is also zero.

If we now include the kinetic term, only the operators of the form

$$\hat{P} = |\alpha\rangle\langle\alpha|, \qquad |\alpha\rangle \equiv e^{\alpha\hat{a}_1^{\dagger} - \overline{\alpha}\hat{a}_1}|0\rangle,$$

corresponding to projections onto a coherent state of the harmonic oscillator, remain as good solitons. To see this we can write the kinetic energy as

$$K = -\frac{1}{2\theta^2}\mathrm{tr}\{[\hat{x}_1, \hat{\Phi}]^2 + [\hat{y}_1, \hat{\Phi}]^2\}, \qquad \hat{x}_1 \equiv \sqrt{\frac{\theta}{2}}(\hat{a}_1 + \hat{a}_1^{\dagger}), \quad \hat{y}_1 \equiv -i\sqrt{\frac{\theta}{2}}(\hat{a}_1 - \hat{a}_1^{\dagger}).$$

For $\hat{P} = |\phi\rangle\langle\phi|$, we find

$$\frac{\theta^2}{\lambda^2}K = \Delta x_1^2 + \Delta y_1^2$$

where

$$\Delta x_1^2 = \langle\phi|\hat{x}_1^2|\phi\rangle - \langle\phi|\hat{x}_1|\phi\rangle^2, \qquad \Delta y_1^2 = \langle\phi|\hat{y}_1^2|\phi\rangle - \langle\phi|\hat{y}_1|\phi\rangle^2$$

are the uncertainties in $\hat{x}_1$ and $\hat{y}_1$. Now we can see that the coherent states, $|\alpha\rangle$, minimize the kinetic energy. This is because:

$$\Delta x_1^2 + \Delta y_1^2 \geq 2\Delta x_1 \Delta y_1 \geq 1,$$

and the equalities hold only for a coherent state. Thus, in the space of all possible unitary transformations, $\hat{U}$, acting on $\hat{\Phi}$, the kinetic energy has flat directions corresponding to translating the brane rigidly in the z_1 direction.

Now, let us add two extra noncommutative directions:

$$x_1 \star y_1 - y_1 \star x_1 = x_2 \star y_2 - y_2 \star x_2 = i\theta.$$

As with z_1, z_2 corresponds to an operator on a Hilbert space $\mathcal{H}_2$. Φ, as a function of x_1, y_1, x_2 and y_2, corresponds to an operator on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. $\mathcal{H}$ has a basis $|N, n\rangle$ defined by $\hat{a}_1^{\dagger}|N, n\rangle = \sqrt{N+1}|N+1, n\rangle$, $\hat{a}_1|N, n\rangle = \sqrt{N}|N-1, n\rangle$, $\hat{a}_2^{\dagger}|N, n\rangle = \sqrt{n+1}|N, n+1\rangle$ and $\hat{a}_2|N, n\rangle = \sqrt{n}|N, n-1\rangle$. This is just the tensor product of the harmonic oscillator eigenstates in each Hilbert space. The soliton described above,

corresponding to a codimension-2 brane with $z_1 = 0$, is now described by $\hat{\Phi} = \lambda \hat{P}_1$, where $\hat{P}_1$ is given by

$$\hat{P}_1 = \sum_{n=0}^{\infty} |0, n\rangle\langle 0, n|. \tag{1}$$

The codimension-2 brane with $z_2 = 0$ is similarly given by $\hat{\Phi} = \lambda \hat{P}_2$,

$$\hat{P}_2 = \sum_{N=0}^{\infty} |N, 0\rangle\langle N, 0|. \tag{2}$$

3.2 Unitary Fluctuations

We now consider the soliton given by $\hat{U}^\dagger \hat{P}_1 \hat{U}$, where $\hat{U}$ is some unitary operator on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We are interested in the kinetic energy as a function of $\hat{U}$. This is more involved than before, so we will work only to second order with

$$\hat{U} = e^{i\epsilon\hat{\Lambda}} = 1 + i\epsilon\hat{\Lambda} + -\frac{1}{2}\epsilon^2\hat{\Lambda}^2 + \mathcal{O}(\epsilon^3)$$

for ϵ real and small and $\hat{\Lambda}$ Hermitian. Define

$$\hat{\Lambda}|0, j\rangle = \sum_{I,i} b_{Ii}^j |I, i\rangle.$$

Following [5], we now obtain the effective Hamiltonian for small fluctuations of the brane. In the operator language, the kinetic energy is:

$$\begin{aligned}
K &= -\frac{1}{2\theta^2}\sum_{k=1}^{2} \text{tr}\{[\hat{x}_k, \hat{\Phi}]^2 + [\hat{y}_k, \hat{\Phi}]^2\} = \frac{1}{\theta^2}\sum_{k=1}^{2} \text{tr}\{[\hat{a}_k, \hat{\Phi}][\hat{\Phi}, \hat{a}_k^\dagger]\} \\
&= \frac{2}{\theta}\text{tr}\{\hat{\Phi}\hat{H}\hat{\Phi} - \sum_{k=1}^{2}(\hat{\Phi}\hat{a}_k\hat{\Phi})(\hat{\Phi}\hat{a}_k^\dagger\hat{\Phi})\}
\end{aligned}$$

where $\hat{H}$ is the harmonic oscillator Hamiltonian,

$$\hat{H} \equiv \sum_{k=1}^{2}\left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right).$$

Any projection operator, $\hat{A}$, such as our soliton, projects onto a subspace, $\mathcal{H}_A$, of the Hilbert space $\mathcal{H}$. Let $|i\rangle$, $i \in \mathcal{S}$, be a basis for $\mathcal{H}_A$. Then we can write the kinetic energy as

$$K = \frac{\lambda^2}{\theta^2}\left(\sum_{i\in\mathcal{S};k=1,2}\langle i|\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger|i\rangle - 2\sum_{i,j\in\mathcal{S};k=1,2}|\langle i|\hat{a}_k^\dagger|j\rangle|^2\right) \tag{3}$$

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This form is sometimes more useful for calculation.

For fluctuations about P_1, to second order in ϵ we obtain:

$$\frac{\theta}{2\lambda^2}K = \sum_k (2k+2) + 2\epsilon^2 \left[\sum_{I\geq 2, j,k\geq 0} (I+k-j)|b_{Ii}^k|^2 - \sum_{j,k\geq 0} |b_{1j}^k|^2 \right.$$

$$- \sum_{j\geq 0}(j+1)\left(1 - \sum_{I\geq 1, i\geq 0}|b_{Ii}^j|^2 - \sum_{I\geq 1, i\geq 0}|b_{Ii}^{j+1}|^2\right)$$

$$\left. - \sum_{I,i,j\geq 1}\sqrt{i(j+1)}\left(\bar{b}_{I,i-1}^j b_{I,i}^{j+1} + b_{I,i-1}^j \bar{b}_{I,i}^{j+1}\right)\right]. \tag{4}$$

This can be rearranged to the positive definite form:

$$\frac{\theta}{2\lambda^2}K = T + 2\epsilon^2 \left[\sum_{I\geq 2; i,k\geq 0} I|b_{Ii}^k|^2 + \sum_{I\geq 1; i\geq 0} i|b_{Ii}^0|^2 \right.$$

$$\left. + \sum_{I\geq 1; j,k\geq 0}\left|\sqrt{k+1}b_{Ij}^k - \sqrt{j+1}b_{I,j+1}^{k+1}\right|^2\right]. \tag{5}$$

Here, T is an infinite constant corresponding to the zero-point energy of the infinite brane. The massless modes must satisfy

$$b_{Ii}^m = 0 \;(\text{for } I \geq 2),$$

$$b_{1,i}^0 = 0 \;(\text{for } i \geq 1), \tag{6}$$

$$\sqrt{m+1}b_{1,n}^m = \sqrt{n+1}b_{1,n+1}^{m+1}. \tag{7}$$

The solution to these constraints is

$$b_{1,n}^m = \begin{cases} 0 & \text{for } m < n \\ \sqrt{\frac{m!}{n!}}c_{m-n} & \text{for } m \geq n \end{cases}$$

where c_m $(m = 0, 1, \ldots)$ are arbitrary constants. Note that when looking at the original form of the kinetic energy (5), we are cancelling two divergent sums. If we demand that all sums converge, the following solution is not legitimate. Throwing caution to the wind, we define the entire holomorphic function $f(\zeta) = \sum_m c_m \zeta^m$. $\hat{\Lambda}$ can then be written as:

$$\hat{\Lambda} = \hat{a}_1^\dagger f(\hat{a}_2) + \hat{a}_1 f(\hat{a}_2)^\dagger + \mathcal{O}(\epsilon^2),$$

and the transformed soliton is:

$$\hat{\Phi} = \lambda\hat{U}^\dagger \hat{P}_1 \hat{U}, \qquad \hat{U} = e^{i\epsilon(\hat{a}_1^\dagger f(\hat{a}_2) + \hat{a}_1 f(\hat{a}_2)^\dagger)} + \mathcal{O}(\epsilon^2).$$

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Physically, this is interpreted as a deformation of the brane from $z_1 = 0$ to $z_1 = \sqrt{2\theta}\epsilon f(\frac{z_2}{\sqrt{2\theta}})$. We can now understand the divergences in this solution as stemming from the fact that a nonconstant entire function cannot be bounded and, as such, these are infinitely large deformations of the brane. If we cut off the sums to force them to be finite, we can still understand these as local approximate zero modes. Another way to understand local behavior is to begin with the equations of motion which follow from the above kinetic energy. This allows us to directly study localized fluctuations. We will examine this further in section (4).

We can rearrange the terms in the kinetic energy into the following (also positive definite) form:

$$\frac{\theta}{2\lambda^2}K = T + 2\epsilon^2 \left[\sum_{I \geq 2, i, k \geq 0} I|b_{Ii}^k|^2 + \sum_{I \geq 1, k \geq 0} k|b_{I0}^k|^2 \right. $$
$$\left. + \sum_{I \geq 1, j, k \geq 0} \left| \sqrt{j+1}b_{Ij}^k - \sqrt{k+1}b_{I,j+1}^{k+1} \right|^2 \right]. \tag{8}$$

Repeating the above analysis, we find that the massless modes for this form of the kinetic energy are

$$b_{1,n}^m = \begin{cases} 0 & \text{for } m > n \\ \sqrt{\frac{n!}{m!}}c_{n-m} & \text{for } m \leq n \end{cases}.$$

Taking again $f(\zeta) = \sum_m c_m \zeta^m$, we obtain

$$\Lambda = z_1^\dagger f(z_2^\dagger) - z_1 f(z_2^\dagger)^\dagger.$$

This corresponds to a deformation of the brane from $z_1 = 0$ to $z_1 = \sqrt{2\theta}\epsilon f(\frac{\bar{z}_2}{\sqrt{2\theta}})$, an anti-holomorphic deformation.

4 Small fluctuations of a flat brane at higher orders

What happens to the zero modes that describe the holomorphic and anti-holomorphic fluctuations at higher orders?

A "classical" 2D (static) membrane in $\mathbb{R}^4$ is described by the equation of motion that states that the area should be minimal under local deformations. At large distances, the solitons in noncommutative field theory also look like 2D membranes, and we will assume

that the curvature, R, of these solitonic membranes is much smaller than the scale set by the noncommutativity, $R \ll \theta^{-2}$. In this section we will set $\theta = \frac{1}{2}$.

These noncommutative solitons differ from the classical membrane in two major ways:

- The antisymmetric 2-form that determines the noncommutativity specifies a preferred complex structure. Thus the $SO(4)$ symmetry of $\mathbb{R}^4$ is broken to $U(2)$. This suggests that deforming a flat soliton by an anti-holomorphic deformation into a curve of the form $z_1 = \epsilon f(\overline{z}_2)$ might not be an exact solution.

- The effective action of the soliton might receive curvature dependent corrections even for a holomorphic deformation $z_1 = \epsilon f(z_2)$.

In this section we will study both these questions. We set

$$\hat{\Phi} \equiv e^{-i\hat{\Lambda}} \hat{P}_1 e^{i\hat{\Lambda}}, \qquad \hat{\Lambda} \equiv \epsilon \hat{a}_1^\dagger \hat{f}(\hat{a}_2, \hat{a}_2^\dagger) + \epsilon \hat{a}_1 \hat{f}(\hat{a}_2, \hat{a}_2^\dagger)^\dagger,$$

and study the corrections to the equations of motion. We continue to work in the approximation that θ is large.

After we find the corrections to the operator $\hat{\Phi}$ in an ϵ expansion, we will translate the operator $\hat{\Phi}$ into a function $\Phi(z_1, z_2, \overline{z}_1, \overline{z}_2)$ via the Weyl transformation:

$$\Phi(z_1, z_2, \overline{z}_1, \overline{z}_2) = \frac{1}{\pi^2} \int \prod_{k=1}^{2} d^2\zeta_k \, e^{i\sum_{k=1}^{2} \zeta_k \overline{z}_k + i\sum_{k=1}^{2} \overline{\zeta}_k z_k} \mathrm{tr}\{e^{-i\sum_{k=1}^{2} \zeta_k \hat{a}_k^\dagger - i\sum_{k=1}^{2} \overline{\zeta}_k \hat{a}_k} \hat{\Phi}\}.$$

We will then solve for the maximum of Φ for a given z_2 so as to find the equation for the curve that is the approximate macroscopic description of the soliton. This is an equation of the form $z_1 = \varphi(z_2, \overline{z}_2)$. To lowest order in ϵ we always obtain $\varphi = \frac{1}{\sqrt{2}} \epsilon f + \mathcal{O}(\epsilon^2)$, where f is the Weyl transform of $\hat{f}$. We will be interested in the higher order corrections.

4.1 The equations of motion

We now describe this procedure in greater detail. We begin by examining the equations of motion. Instead of writing the equations of motion for $\hat{\Lambda}$, it will be more convenient to write the equations for $\hat{\Phi}$ directly. Starting with

$$\Phi_0 = 2e^{-|z_1|^2} \implies \hat{\Phi}_0 = \sum_{n=0}^{\infty} |0, n\rangle\langle 0, n|,$$

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we take the unitary operator $\hat{U} \equiv e^{i\hat{\Lambda}}$ and define $\hat{\Phi} = \hat{U}^\dagger \hat{\Phi}_0 \hat{U}$. The equations of motion are obtained by minimizing the kinetic energy that is proportional to:

$$\sum_{i=1}^{2} \text{tr}\{[\hat{a}_i, \hat{\Phi}][\hat{a}_i^\dagger, \hat{\Phi}]\}$$

with respect to $\hat{\Lambda}$. However, it will turn out to be more convenient to write an equation of motion for $\hat{\Phi}$. We must minimize K under the condition $\hat{\Phi} \star \hat{\Phi} = \hat{\Phi}$, so we insert a Lagrange multiplier, χ, to enforce the constraint. This gives:

$$\Delta\hat{\Phi} = \chi \star \hat{\Phi} + \hat{\Phi} \star \chi - \chi \tag{9}$$

where

$$\Delta\hat{\Phi} \equiv \sum_{i=1}^{2} [\hat{a}_i, [\hat{a}_i^\dagger, \hat{\Phi}]].$$

In general, a Hermitian operator, $\hat{O}$, that can be written as

$$\hat{O} = \chi \star \hat{\Phi} + \hat{\Phi} \star \chi - \chi$$

satisfies

$$\hat{O} \star \hat{\Phi} = \hat{\Phi} \star \hat{O}.$$

Alternatively, given an operator, $\hat{O}$, that commutes with $\hat{\Phi}$ we can satisfy (9) by choosing

$$\chi = 2\hat{\Phi} \star \hat{O} - \hat{O}.$$

Thus the equations of motion are equivalent to:

$$\hat{\Phi} \star \hat{\Phi} = \hat{\Phi}, \qquad [\Delta\hat{\Phi}, \hat{\Phi}] = 0. \tag{10}$$

4.2 Anti-holomorphic fluctuations

In order to further study anti-holomorphic fluctuations, we take:

$$\hat{\Lambda} = \rho e^{i\phi}\hat{a}_1^\dagger \hat{a}_2^\dagger + \rho e^{-i\phi}\hat{a}_1 \hat{a}_2,$$

where ρ and ϕ are real. The Weyl transform of $e^{i\hat{\Lambda}}\hat{P}_1 e^{-i\hat{\Lambda}}$ is

$$\Phi = 2e^{-|z_1 \cosh\rho - i\bar{z}_2 e^{i\phi} \sinh\rho|^2}. \tag{11}$$

The operator, $e^{i\hat{\Lambda}}$, generates an $SO(4)$ rotation of $\mathbb{R}^4$ that is not in $U(2) \subset SO(4)$. Therefore, the maxima of Φ correspond to a plane that is not a holomorphic curve in the preferred complex structure that is determined by the noncommutativity. However, it is easy to see that $\hat{\Phi}$ is still a solution of the equations of motion (10). This is because the Laplacian operator, Δ, is $SO(4)$ invariant and not just $U(2)$ invariant.

The kinetic energy density along the soliton given by (11) is independent of ρ. However, the "width" of the soliton is proportional to $1/\cosh 2\rho$. So, *microscopically*, the solitons that correspond to non holomorphic curves differ from the holomorphic ones in that they are "thinner". It is amusing to note that for $\rho = \infty$, the curve is $z_1 = e^{i\phi}\overline{z}_2$, and the width of the soliton is zero. However, *macroscopically*, all the planar solitons have the same energy density and the microscopic distinction between different directions probably disappears because the $SO(4)$ symmetry is restored. It would be interesting to confirm this with scattering calculations.

4.3   Curvature

Finally, we would like to study higher order corrections to a holomorphic deformation. For this, we take:

$$\hat{\Lambda}_1 = \epsilon(\beta\hat{a}_1^\dagger \hat{a}_2^2 + \overline{\beta}\hat{a}_1 \hat{a}_2^{\dagger 2})$$

and define $\hat{\Phi} = e^{-i\hat{\Lambda}}\hat{P}_1 e^{i\hat{\Lambda}}$. This corresponds to placing the brane along the curve $z_1 = \epsilon\beta\zeta_2^2$. We wish to calculate:

$$\hat{\Xi}_1 \equiv [\Delta\hat{\Phi}, \hat{\Phi}].$$

For these values, we have:

$$\hat{\Xi}_1 = -4\epsilon^3(\beta^2\overline{\beta}\hat{a}_1^\dagger \hat{P}_1 \hat{a}_2^2 - \beta\overline{\beta}^2 \hat{P}_1 \hat{a}_1 \hat{a}_2^{\dagger 2}) + \mathcal{O}(\epsilon^4).$$

In order to satisfy the equations of motion, this should be zero. Towards that end, we can cancel the ϵ^3 term by augmenting $\hat{\Lambda}_1$ to:

$$\hat{\Lambda}_2 \equiv \epsilon(\beta\hat{a}_1^\dagger \hat{a}_2^2 + \overline{\beta}\hat{a}_1 \hat{a}_2^{\dagger 2}) - \frac{4}{3}\epsilon^3(\beta^2\overline{\beta}\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2^3 + \overline{\beta}^2\beta\hat{a}_1 \hat{a}_2^{\dagger 3} \hat{a}_2).$$

This should not be considered a modification of the equation of motion for the fluctuation $f(\hat{a}_2, \hat{a}_2^\dagger)$ because, at the current order of approximation in ϵ, near $z_2 = 0$, the maximum of the Weyl transform of $\hat{\Phi}$ still defines the curve $z_1 = \epsilon\beta z_2^2$, as we will soon see.

We can continue this procedure to higher orders. At the nth order we will have an approximate $\hat{\Lambda}_n$ that is correct up to (but not including) $\mathcal{O}(\epsilon^{n+2})$. We can then calculate $\hat{\Phi} = e^{-i\hat{\Lambda}_n}\hat{P}_1 e^{i\hat{\Lambda}_n}$ and define $\hat{\Xi}_n \equiv [\Delta\hat{\Phi}, \hat{\Phi}]$ which will be of order $\mathcal{O}(\epsilon^{n+3})$. We can then try to correct $\hat{\Lambda}_n$ by a Hermitian operator that will cancel $\hat{\Xi}_n$ up to the $(n+3)^{rd}$ order. To find this we set $\hat{\Lambda}_{n+1} = \hat{\Lambda}_n + \delta\hat{\Lambda}$. We then write the linearized equation for $\delta\hat{\Phi} \equiv i[\hat{\Phi}_0, \delta\hat{\Lambda}] + \mathcal{O}(\epsilon^{n+4})$. It is:

$$[\Delta\delta\hat{\Phi}, \hat{\Phi}_0] + [\Delta\hat{\Phi}_0, \delta\hat{\Phi}] = -\Xi_n. \tag{12}$$

Here we can set:

$$\hat{\Phi}_0 = \hat{P}_1, \qquad \Delta\hat{\Phi}_0 = \hat{a}_1^\dagger \hat{P}_1 \hat{a}_1 - \hat{P}_1.$$

The equation (12) has solutions that are unique up to the zero modes found above. These are:

$$\delta\hat{\Lambda} = \sum_{n=0}^{\infty} C_n \hat{a}_1^\dagger \hat{a}_2^n + \sum_{n=0}^{\infty} \overline{C}_n \hat{a}_1 \hat{a}_2^{\dagger n} + \sum_{n=0}^{\infty} C'_n \hat{a}_1^\dagger \hat{a}_2^{\dagger n} + \sum_{n=0}^{\infty} \overline{C}'_n \hat{a}_1 \hat{a}_2^n.$$

We make sure that $\hat{\Lambda}$ does not contain these terms except for the term $\epsilon\beta\hat{a}_2^2$ that we began with.

At the next order we define $\hat{\Phi} = e^{-i\hat{\Lambda}_1}\hat{\Phi}_0 e^{i\hat{\Lambda}_1}$ and calculate $\Xi_2 \equiv [\Delta\hat{\Phi}, \hat{\Phi}]$. We find:

$$\Xi_2 = \frac{2}{3}\epsilon^4(\beta^3\overline{\beta}\hat{a}_1^{\dagger 2}\hat{P}_1\hat{a}_2^4 - \beta\overline{\beta}^3\hat{P}_1\hat{a}_1^2\hat{a}_2^{\dagger 4}) + \mathcal{O}(\epsilon^5).$$

We can correct this by augmenting $\hat{\Lambda}$ to:

$$\begin{aligned}
\hat{\Lambda}_3 &\equiv \epsilon(\beta\hat{a}_1^\dagger \hat{a}_2^2 + \overline{\beta}\hat{a}_1\hat{a}_2^{\dagger 2}) - \frac{4}{3}\epsilon^3(\beta^2\overline{\beta}\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2^3 + \overline{\beta}^2\beta\hat{a}_1\hat{a}_2^{\dagger 3}\hat{a}_2) \\
&\quad - \frac{i}{3}\epsilon^4(\beta^3\overline{\beta}\hat{a}_1^{\dagger 2}\hat{a}_2^4 - \overline{\beta}^3\beta\hat{a}_1^2\hat{a}_2^{\dagger 4}).
\end{aligned}$$

Continuing this procedure we find that, up to $\mathcal{O}(\epsilon^8)$ terms, the following is a solution of the equations of motion:

$$\begin{aligned}
\hat{\Lambda} &= \epsilon\beta\hat{a}_1^\dagger \hat{a}_2^2 + \epsilon\overline{\beta}\hat{a}_1\hat{a}_2^{\dagger 2} \\
&\quad - \left(\frac{4}{3}\epsilon^3\beta^2\overline{\beta} - \frac{56}{15}\epsilon^5\beta^3\overline{\beta}^2 + \frac{3872}{315}\epsilon^7\beta^4\overline{\beta}^3\right)\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2^3 \\
&\quad - \left(\frac{4}{3}\epsilon^3\beta\overline{\beta}^2 - \frac{56}{15}\epsilon^5\beta^2\overline{\beta}^3 + \frac{3872}{315}\epsilon^7\beta^3\overline{\beta}^4\right)\hat{a}_1\hat{a}_2^{\dagger 3}\hat{a}_2 \\
&\quad - \left(\frac{i}{3}\epsilon^4\beta^3\overline{\beta} - \frac{139i}{45}\epsilon^6\beta^4\overline{\beta}^2\right)\hat{a}_1^{\dagger 2}\hat{a}_2^4 + \left(\frac{i}{3}\epsilon^4\beta\overline{\beta}^3 - \frac{139i}{45}\epsilon^6\beta^2\overline{\beta}^4\right)\hat{a}_1^2\hat{a}_2^{\dagger 4}
\end{aligned}$$

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$$+\left(\frac{191}{45}\epsilon^5\beta^3\overline{\beta}^2 - \frac{40121}{945}\epsilon^7\beta^4\overline{\beta}^3\right)\hat{a}_1^\dagger\hat{a}_2^{\dagger 2}\hat{a}_2^4$$

$$+\left(\frac{191}{45}\epsilon^5\beta^2\overline{\beta}^3 - \frac{40121}{945}\epsilon^7\beta^3\overline{\beta}^4\right)\hat{a}_1\hat{a}_2^{\dagger 4}\hat{a}_2^2$$

$$+\frac{142i}{45}\epsilon^6\beta^4\overline{\beta}^2\hat{a}_1^{\dagger 2}\hat{a}_2^\dagger\hat{a}_2^5 - \frac{142i}{45}\epsilon^6\beta^2\overline{\beta}^4\hat{a}_1^2\hat{a}_2^{\dagger 5}\hat{a}_2$$

$$-\frac{176}{945}\epsilon^7\beta^5\overline{\beta}^2\hat{a}_1^{\dagger 3}\hat{a}_2^6 - \frac{176}{945}\epsilon^7\beta^2\overline{\beta}^5\hat{a}_1^3\hat{a}_2^{\dagger 6}$$

$$-\frac{17162}{945}\epsilon^7\beta^4\overline{\beta}^3\hat{a}_1^\dagger\hat{a}_2^{\dagger 3}\hat{a}_2^5 - \frac{17162}{945}\epsilon^7\beta^3\overline{\beta}^4\hat{a}_1\hat{a}_2^{\dagger 5}\hat{a}_2^3 + \mathcal{O}(\epsilon^8).$$

Substituting this into $\hat{\Phi}$ and performing a Weyl transformation, we can find the expression for the field $\Phi(z_1, z_2, \overline{z}_1, \overline{z}_2)$. Since the expression is rather long, we will only present the leading terms below:

$$
\begin{aligned}
\Phi &= 2e^{-|z_1|^2+\psi}, \\
\psi &= \frac{\sqrt{2}}{2}i\epsilon(\beta z_2^2\overline{z}_1 - \overline{\beta}z_1\overline{z}_2^2) + \epsilon^2(-|\beta|^2 + |\beta|^2 z_1\overline{z}_1 + 2|\beta|^2 z_1 z_2\overline{z}_1\overline{z}_2 - \frac{1}{2}|\beta|^2 z_2^2\overline{z}_2^2) \\
&\quad + i\sqrt{2}\epsilon^3(-\frac{1}{6}\beta|\beta|^2 z_2^2\overline{z}_1 + \beta|\beta|^2 z_1 z_2^2\overline{z}_1^2 - \beta|\beta|^2 z_2^3\overline{z}_1\overline{z}_2 \\
&\quad + \frac{1}{6}\overline{\beta}|\beta|^2 z_1\overline{z}_2^2 - \overline{\beta}|\beta|^2 z_1^2\overline{z}_1\overline{z}_2^2 + \overline{\beta}|\beta|^2 z_1 z_2\overline{z}_2^3) \\
&\quad + \epsilon^4(\beta^2|\beta|^2 z_2^4\overline{z}_1^2 + \overline{\beta}^2|\beta|^2 z_1^2\overline{z}_2^4 - \frac{17}{6}|\beta|^4 + \frac{1}{3}|\beta|^2 z_1\overline{z}_1 \\
&\quad + |\beta|^2|z_1|^4 + 6|\beta|^2|z_2|^2 - \frac{16}{3}|\beta|^4|z_1|^2|z_2|^2 + 4|\beta|^4|z_1|^4|z_2|^2 \\
&\quad - \frac{1}{6}|\beta|^4|z_2|^4 - 8|\beta|^4|z_1|^2|z_2|^4 + |\beta|^4|\overline{z}_2|^6) + \mathcal{O}(\epsilon^5). \quad\quad (13)
\end{aligned}
$$

We can now look for the maximum of Φ. This will approximately outline the curve that a macroscopic observer would see as a 2-brane. The minimum of the exponent is at:

$$z_1 = \frac{i}{\sqrt{2}}\epsilon\beta z_2^2(1 + \frac{2}{3}\epsilon^2|\beta|^2 + \frac{188}{15}\epsilon^4|\beta|^4 + \frac{8956}{315}\epsilon^6|\beta|^6 - 96\epsilon^6|\beta|^6|z_2|^2) + \mathcal{O}(\epsilon^8). \quad (14)$$

We see that up to order $\mathcal{O}(\epsilon^6)$, all the corrections can be interpreted as a renormalization of β. At large scale there are no corrections to the parabolic shape of the graph of the brane. In particular, the curve is still analytic. The first deviation from analyticity occurs at order $\mathcal{O}(\epsilon^7)$ because of the appearance of the $z_2^2|z_2|^2$ term. To this order φ is no longer harmonic and instead satisfies:

$$z_1 = \varphi(z_2, \overline{z}_2), \quad\quad \partial\overline{\partial}\varphi = -18(\partial\varphi)^2(\partial^2\varphi)^2(\overline{\partial}^2\overline{\varphi})^3.$$

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4.4 Region of Validity of the Approximation

We have started to construct, order by order, a solution that looks macroscopically near the origin like the curve $z_1 = \epsilon\beta z_2^2$. By "macroscopically" we mean that distances are larger than the noncommutativity scale. We have set the noncommutativity scale to 1 here, so we require that the solution be valid not only for $z_1, z_2 \sim 0$ but also for $|z_1|, |z_2| \gg 1$! On the other hand, we wish to assume that the curvature of the curve is small at the origin and, as far as the geometry of the curve goes, we are in the vicinity of the origin. Quantitatively, this requires that $|\epsilon\beta z_2| \ll 1$. Looking at the solution, (13), we see that the order of magnitude of the $\mathcal{O}(\epsilon^{2n})$ in $\hat{\Lambda}$ is smaller by a factor of $\epsilon\beta z_1$ from the $\mathcal{O}(\epsilon^{2n-1})$ terms and the $\mathcal{O}(\epsilon^{2n+1})$ terms are smaller by a factor of $\epsilon^2|\beta|^2|z_2|^2$ from the $\mathcal{O}(\epsilon^{2n-1})$ terms. So, the approximation is within the required region of validity.

Note, however, that in the region of validity of the calculation, ie, $\epsilon|\beta z_2| \ll 1$, the correction to z_1 in (14) is smaller than 1, and thus is actually microscopic.

5   Intersecting D2-Branes

5.1   Construction of the Intersecting Soliton

In the previous section, we constructed a D2-brane at $z_1 = 0$ as $\hat{\Phi}_1 = \lambda\hat{P}_1$ and a D2-brane at $z_2 = 0$ as $\hat{\Phi}_2 = \lambda\hat{P}_2$. We now wish to find a soliton $\hat{\Phi} = \lambda\hat{P}$ which asymptotically looks like $\hat{\Phi}_1 + \hat{\Phi}_2$. This is straightforward. We define

$$\hat{P}_\eta = \hat{P}_1 + \hat{P}_2 - \eta\hat{P}_1\hat{P}_2, \qquad \hat{\Phi}_\eta = \lambda\hat{P}_\eta$$

This will be a projection operator for $\eta = 1$ or $\eta = 2$. To distinguish between the two solutions, we need to calculate their kinetic energy, (3). While each solution has an infinite kinetic energy because of its infinite extent, the difference is finite and easy to calculate:

$$K(\hat{\Phi}_{\eta=2}) - K(\hat{\Phi}_{\eta=1}) = \frac{4\lambda^2}{\theta}.$$

Thus, $\eta = 1$ corresponds to the solution with the lower kinetic energy. We propose that this solution corresponds to two intersecting branes. The $\eta = 2$ solution is similar, but it has a

'hole' attached at the intersection:

$$\hat{P}_{\eta=2} = \hat{P}_{\eta=1} - \hat{P}_1\hat{P}_2.$$

In a sense, it is as if a 0-brane (represented by $\hat{P}_1\hat{P}_2$) had been removed. This solution will turn out to be unstable to small unitary perturbations.

5.2 Fluctuations

We now wish to repeat the calculation of the effective Hamiltonian for small fluctuations of the two intersecting branes. Consider the fluctuation given by $\hat{U}^\dagger \hat{P}_\eta \hat{U}$, where $\hat{U}$ is again a unitary operator on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. As before, let

$$\hat{U} = e^{i\epsilon\hat{\Lambda}} = 1 + i\epsilon\hat{\Lambda} - \frac{1}{2}\epsilon^2\hat{\Lambda}^2 + \mathcal{O}(\epsilon^3) \tag{15}$$

with ϵ real and small and $\hat{\Lambda}$ hermitian. One can calculate the kinetic energy for this soliton to second order in ϵ. This is most conveniently done from equation (3).

In the $\eta = 1$ case, we define

$$
\begin{aligned}
\hat{\Lambda}|0,j\rangle &= \sum_{I,i} b_{Ii}^{j}|I,i\rangle, \quad I,i,j \geq 1, \\
\hat{\Lambda}|J,0\rangle &= \sum_{I,i} c_{Ii}^{J}|I,i\rangle, \quad I,i,J \geq 1 \\
\hat{\Lambda}|0,0\rangle &= \sum_{I,i} d_{Ii}|I,i\rangle. \quad I,i \geq 1
\end{aligned}
\tag{16}
$$

After consolidation of terms, (3) becomes:

$$
\begin{aligned}
\frac{\theta}{2\lambda^2}K_{\eta=1} &= \frac{\theta}{2\lambda^2}K(\hat{\Phi}_{\eta=1}) + 2\epsilon^2 \Bigg[\sum_{J \geq 2, j, k \geq 1} J|b_{Jj}^{k}|^2 + \sum_{j \geq 2, J, K \geq 1} j|c_{Jj}^{K}|^2 \\
&+ \sum_{J,j,k \geq 1} \left|\sqrt{k+1}b_{Jj}^{k} - \sqrt{j+1}b_{J,j+1}^{k+1}\right|^2 + \sum_{J,j,K \geq 1} \left|\sqrt{K+1}c_{Jj}^{k} - \sqrt{J+1}c_{J+1,j}^{K+1}\right|^2 \\
&+ \sum_{J \geq 2, j \geq 1} J|d_{Jj}|^2 + \sum_{j \geq 2, J \geq 1} j|d_{Jj}|^2 \\
&+ \sum_{J,j \geq 1} \left|d_{Jj} - \sqrt{j+1}b_{J,j+1}^{1}\right|^2 + \sum_{J,j \geq 1} \left|d_{Jj} - \sqrt{J+1}c_{J+1,j}^{1}\right|^2 \Bigg]
\end{aligned}
\tag{17}
$$

where $K(\hat{\Phi}_{\eta=1})$ is the (infinite) energy of an undistorted soliton discussed in previous subsection.

Using the same procedure as before, we obtain the following zero modes:

$$
b_{1,n}^m = \begin{cases}
0 & \text{for } m+1 < n \\
\frac{d_{11}}{\sqrt{n}} & \text{for } m+1 = n \\
d_{10} & \text{for } m = n \\
\sqrt{\frac{m!}{n!}}\, p_{m-n} & \text{for } m > n
\end{cases}
$$

and

$$
c_{N,1}^M = \begin{cases}
0 & \text{for } M+1 < N \\
\frac{d_{11}}{\sqrt{N}} & \text{for } M+1 = N \\
d_{01} & \text{for } M = N \\
\sqrt{\frac{M!}{N!}}\, q_{M-N} & \text{for } M > N
\end{cases}
$$

with d_{Jj} for all $J, j \geq 2$ equal to zero. The p's and q's are arbitrary constants. These can be used to define two entire holomorphic functions $f_1(\zeta) = \sum_m p_m \zeta^m$ and $f_2(\zeta) = \sum_M q_M \zeta^M$. These zero modes, just as for a single brane, correspond to deformations of the two branes: $z_1' = \epsilon f_1(z_2)$ and $z_2' = \epsilon f_2(z_1)$. As in the case of a single brane, the terms in the kinetic energy can be rearranged to make apparent the antiholomorphic deformations.

A new phenomenon is the mode corresponding to a non-zero d_{11} together with $b_{1,k+1}^k = d_{11}(k+1)^{-1/2}$ and $c_{K+1,1}^K = d_{11}(K+1)^{-1/2}$ so that the terms in kinetic energy that are differences vanish. This mode might be thought of as

$$
\Lambda \sim \frac{\alpha}{z_1 z_2} + \frac{\bar{\alpha}}{z_1^\dagger z_2^\dagger}.
$$

This is a complex mode (two real modes) corresponding to the extra degrees of freedom living on the intersection of the two branes.

We now consider the case of $\eta = 2$. Here, (3) reduces to

$$
\begin{aligned}
\frac{\theta}{2\lambda^2} K_{\eta=1} \;=\; & \frac{\theta}{2\lambda^2} K(\hat{\Phi}_{\eta=1}) + 2\epsilon^2 \Bigg[\sum_{J\geq 2, j,k\geq 1} J|b_{Jj}^k|^2 + \sum_{j\geq 2, J,K\geq 1} j|c_{Jj}^K|^2 \\
& + \sum_{J,j,k\geq 1} \left| \sqrt{k+1}\, b_{Jj}^k - \sqrt{j+1}\, b_{J,j+1}^{k+1} \right|^2 + \sum_{J,j,K\geq 1} \left| \sqrt{K+1}\, c_{Jj}^k - \sqrt{J+1}\, c_{J+1,j}^{K+1} \right|^2 \\
& + \sum_{J,k\geq 2}(J+k-1)|b_{Jk}^1|^2 + \sum_{J\geq 2}(J-1)|b_{J+1,1}^1|^2 + \sum_{k\geq 2}(k-1)|b_{1,k}^1|^2 \\
& + \sum_{j,K\geq 2}(j+K-1)|c_{Kj}^1|^2 + \sum_{j\geq 2}(j-1)|c_{1,j+1}^1|^2 + \sum_{K\geq 2}(K-1)|c_{K,1}^1|^2
\end{aligned}
$$

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$$+ \sum_{k \geq 2} k |b_{00}^{k+1}|^2 + \sum_{K \geq 2} K |c_{00}^{K+1}|^2 + \sum_{K \geq 2} |b_{K1}^1 + \bar{c}_{00}^K|^2 + \sum_{k \geq 2} |c_{1k}^1 + \bar{b}_{00}^k|^2$$

$$- |b_{11}^1 + \bar{c}_{00}^1|^2 - |c_{11}^1 + \bar{b}_{00}^1|^2 \Bigg]. \tag{18}$$

The zero modes, which we will not write out explicitly, include our familiar entire holomorphic and anti-holomorphic deformations of the branes. More importantly, we now have unstable modes given by $b_{11}^1 + \bar{c}_{00}^1 \neq 0$ and $c_{11}^1 + \bar{b}_{00}^1 \neq 0$ together with extra elements so that the terms that are differences are zero. These two modes correspond to moving the aforementioned 'hole' away from the intersection along either of the two branes. We also note that the above effective Hamiltonian has an additional zero mode given by $\Lambda = \alpha (\hat{a}_1^\dagger)^2 + \bar{\alpha} (\hat{a}_1)^2$ (and similarly for $\hat{a}_2$), which corresponds to distorting the shape of the hole from the gaussian ground state of a harmonic oscillator into a squeezed state.

6 Extensions and Discussion

6.1 Multiple Branes

Our construction for two intersecting D2-branes can easily be extended to a larger number of branes.

For example, let $\hat{P}_1^K$ be a projection operator corresponding to a stack of K branes at $z_1 = 0$ and $\hat{P}_2^L$ be a projection operator corresponding to a stack of L branes at $z_2 = 0$. This means that $\hat{P}_1^K$ can be written as a sum of K projection operators

$$\hat{P}_1^K = \sum_{i=1}^K \hat{p}_1^i$$

with $\hat{p}_1^i \hat{p}_1^j = \delta^{ij} \hat{p}_1^i$, each $\hat{p}_1^i$ being a projection operator for a single brane. Similarly,

$$\hat{P}_2^L = \sum_{i=1}^L \hat{p}_2^i.$$

Now, any operator of the form

$$\hat{P}_1^K + \hat{P}_2^L - \sum_{i=1}^K \sum_{j=1}^L \eta_{ij} \hat{p}_1^i \hat{p}_2^j$$

for $\eta_{ij} = 1, 2$ corresponds to an intersection of these two stacks.

As another example, let us take $\mathbb{R}^6$, i.e. three complex dimensions. Let $\hat{P}_{12}$ correspond to a codimension-2 brane at $z_3 = 0$, $\hat{P}_{23}$ correspond to a codimension-2 brane at $z_1 = 0$ and $\hat{P}_{31}$ correspond to a codimension-2 brane at $z_2 = 0$. Then it can be checked that

$$\hat{P}_{12} + \hat{P}_{23} + \hat{P}_{31} - \eta_{12}\hat{P}_{23}\hat{P}_{31} - \eta_{23}\hat{P}_{12}\hat{P}_{31} - \eta_{31}\hat{P}_{12}\hat{P}_{23} + (\eta_{12} + \eta_{23} + \eta_{31} - \eta - 1)\hat{P}_{12}\hat{P}_{23}\hat{P}_{31}$$

is a projection operator corresponding to the intersection of all three branes at a point, provided we set $\eta_{12}, \eta_{23}, \eta_{31}, \eta \in \{1, 2\}$. It is straightforward, if a bit tedious, to extend this to any number of branes.

6.2 Discussion

In this paper we have found a solution that describes two intersecting 2-brane solitons in a field theory on a noncommutative $\mathbb{R}^4$ in the large noncommutativity limit. We studied the zero modes of the solution. We found a zero mode that is reminiscent of the zero mode of two intersecting D2-branes that corresponds to a deformation into an irreducible curve. It would be interesting to examine whether this zero mode receives a potential at higher orders or whether it is an exact flat direction. Because we have seen that simple holomorphic deformations are no longer flat at sufficiently high orders, the latter possibility seems unlikely. Here "higher-order" could have several meanings. First there is the expansion of the classical action, still in the large noncommutativity limit. This expansion parameter is the ϵ in equation (15). On top of that, there are the expansions in the noncommutativity parameter and the quantum fluctuations. In these notes we have not attempted to include either of those.

It has recently been shown [12, 10, 11] that, in the situation of a noncommutative $U(\infty)$ gauge theory, one can cancel the kinetic term in the action through a suitable configuration of the gauge fields. The soliton configurations discussed in this paper are easily realizable in the schemes of the referenced papers. It might be interesting to compute the actions for perturbations of the fields as in [12, 10]. However, because one can always find a gauge field to cancel the kinetic term of a given soliton and because the projection operators here are

halving projections[4], it should be possible to continuously interpolate through conjugation with unitary operators between these soliton configurations and other configurations that are described by halving projections. This includes, in the case of four noncommutative directions, any number of branes in any given direction. It is not immediately clear to us what this means.

Acknowledgments

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